Sofic groups and profinite topology on free groups *

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Abstract

We give a definition of weakly sofic groups (w-sofic groups). Our definition is a rather natural extension of the definition of sofic groups where instead of the Hamming metric on symmetric groups we use general bi-invariant metrics on finite groups. The existence of non w-sofic groups is equivalent to a profinite topology property of products of conjugacy classes in free groups.

Key words: Sofic groups, profinite topology, conjugacy classes, free groups.
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1 Introduction

The notion of sofic groups was introduced in [9,19] in relation with the problem "If for every group G the injectivity of cellular automata over G implies their surjectivity?" due to Gottschalk [8]. The problem is still open and the class of sofic groups is the largest class of groups for which the problem is proved to have a positive solution [19], see also [3,4]. Sofic groups turned out to be interesting from other points of view. For example, the "Connes Embedding Conjecture" and "Determinant Conjecture" was proven for sofic groups, [6]. The class of sofic groups is closed with respect to various group-theoretic constructions: direct products, some extensions, etc., see [6,7]. It is still an open question if there exists a non sofic group.

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In the article we introduce the apparently more general class of weakly sofic groups (w-sofic groups) (a sofic group is a w-sofic group). We prove that the notion of w-sofic group is closely related to some properties of the profinite topology on free groups. The profinite topology on free groups (a base of neighborhoods of the unit element consists of the normal subgroups of finite index) draw considerable attention and one of the remarkable results here is that a product of a finite number of finitely generated subgroups is closed in the profinite topology (as well as in the pro-p topology) [2,11,18].

The question related to w-sofic groups is about the closure of products of conjugacy classes in the profinite topology.

To finish this section, let us list some notation, that will be used throughout the article. S_n will denote the symmetric group on n elements (the group of all permutations of a finite set $[n] = \{1, ..., n\}$). On S_n we define the normalized Hamming metric $h_n(f, g) = \frac{|\{a:(a)f \neq (a)g\}|}{n}$. It is easy to check that $h_n(\cdot, \cdot)$ is a bi-invariant metric on S_n [5]. Sometimes we will omit the subscript n in h_n. For a group G let e_G denote the unit element of G. For $A, B \subseteq G$ let, as usual, $AB = \{xy \mid x \in A, \ y \in B\}$. $X < G$ and $X \lhd G$ will denote ”X is a subgroup of G” and ”X is a normal subgroup of G”, respectively. For $g \in G$ let $[g]^G$ denote the conjugacy class of g in G. For $g \in G$ and $N < G$ let $[g]_N = gN$ denote the left N-coset of g.

2 Profinite topology

Closed sets in the profinite (the pro-p) topologies can be characterized by the following separability property: a set $X \subseteq G$ is closed in the profinite (the pro-p) topology iff for any $g \notin X$ there exists a homomorphism ϕ from G to a finite group (a finite p-group) such that $\phi(g) \notin \phi(X)$. It is clear, that for a closed $X \subset G$ and $g_1, g_2, ..., g_k \notin X$ there exists a homomorphism ϕ (to a finite group or to a finite p-group, depending on the topology considered) such that $\phi(g_1), \phi(g_2), ..., \phi(g_k) \notin \phi(X)$. (Just take $\ker \phi = \cap \ker \phi_i$, where $\phi_i(g_i) \notin \phi_i(X)$). By the same way one can characterize the closure in the profinite (the pro-p) topology. This characterization will be used in the proof of Theorem 4.3.

In the present article we are interested in the profinite topology on a finitely generated free group F. It is known, that $[g]^F$ is closed in the pro-p (and the profinite) topology for any $g \in F$, (a conjugacy class in a free group is separable by homomorphisms to finite p-groups, see [13]). Up to our knowledge it is an open question if a product of several conjugacy classes $[g_1]^F[g_2]^F...[g_k]^F$ is closed in the profinite topology in a free group F. For pro-p topology the answer is known: in the 2-generated free group $F = \langle x, y \rangle$ there exist elements

g_1 and g_2 such that $[g_1]^F[g_2]^F$ is not closed in any pro-p topology, even more, its closure is not contained in $N(g_1, g_2)$, the normal subgroup generated by g_1, g_2, see [12]. Precisely, in [12] the following is proven. If $g_1 = x^{-2}y^{-3}$, $g_2 = x^{-2}(xy)^5$ and $a = xy^2$, then $a \notin N(g_1, g_2)$. On the other hand, for any i, one has $a \equiv w_i^{-1}g_1w_iv_i^{-1}g_2v_i \mod F_i$ for some $w_i, v_i \in F$, where $F_0 = F$ and $F_{i+1} = [F_i, F]$. So, for any homomorphism ϕ from F to a nilpotent group $\phi(a) \in \phi([g_1]^F[g_2]^F)$ (it must be $\phi(F_i) = \{e\}$ for some i). Since a finite p-group is nilpotent, a belongs to the closure of $[g_1]^F[g_2]^F$ in any pro-p topologies, but $a \notin N(g_1, g_2)$. So, the same statement may be valid for the profinite topology. Let F be a free group and $X \subseteq F$. Let us denote the closure of X in the profinite topology on F by $\overline{X}$.

Conjecture 2.1 *For a finitely generated free group F, there exists a sequence* $g_1, g_2, ..., g_k \in F$ *such that*

$$\overline{[g_1]^F[g_2]^F...[g_k]^F} \not\subseteq N(g_1, g_2, ..., g_k).$$

We hope, that some techniques of [14,15,16] could be useful for resolving the conjecture.

3 Sofic groups

Definition 3.1 *Let H be a finite group with a bi-invariant metric d. Let G be a group, $\Phi \subseteq G$ be a finite subset, $\epsilon > 0$, and $\alpha > 0$. A map $\phi : \Phi \to H$ is said to be a (Φ, ϵ, α)-homomorphism if:*

(1) For any two elements $a, b \in \Phi$, with $a \cdot b \in \Phi$, $d(\phi(a)\phi(b), \phi(a \cdot b)) < \epsilon$
(2) If $e_G \in \Phi$, then $\phi(e_G) = e_H$
(3) For any $a \neq e_G$, $d(\phi(a), e_H) > \alpha$

Definition 3.2 *The group G is sofic if there exists $\alpha > 0$ such that for any finite set $\Phi \subseteq G$, for any $\epsilon > 0$ there exists a (Φ, ϵ, α)-homomorphism to a symmetric group S_n with the normalized Hamming metric h.*

Let us give an equivalent definition which is more widely used, see [6,7].

Definition 3.3 *The group G is sofic if for any finite set $\Phi \subseteq G$, for any $\epsilon > 0$ there exists a $(\Phi, \epsilon, 1 - \epsilon)$-homomorphism to a symmetric group S_n with the normalized Hamming metric h.*

The following proposition, in fact, is contained in [17].

Proposition 3.4 *The definitions 3.2 and 3.3 are equivalent.*

PROOF. A bijection $[n^2] \leftrightarrow \{(i, j) \; : \; i, j \in [n]\}$ naturally defines an injective inclusion $S_n \times S_n \to S_{n^2}$. ($(i, j)f \times g = ((i)f, (j)g)$). One can check that if $f, g \in S_n$, then $(1 - h_{n^2}(g \times g, f \times f)) = (1 - h_n(f, g))^2$. So, if $\phi : G \to S_n$ is a $(\Phi, \epsilon_0, \alpha_0)$-homomorphism, then $\phi \times \phi : G \to S_n \times S_n < S_{n^2}$ is a $(\Phi, \epsilon_1, \alpha_1)$-homomorphism with $\epsilon_1 = 2\epsilon - \epsilon^2$ and $\alpha_1 = 2\alpha - \alpha^2$. Now repeating this operation one can make α_n as close to 1 as one wants, then choose ϵ_0 such that ϵ_n is as small as one wants. (We suppose that $0 < \epsilon < \alpha \leq 1$.) $\quad\square$

4   w-sofic groups and profinite topology

Definition 3.2 of sofic groups appeals to the following generalization:

Definition 4.1 *A group G is called w-sofic if there exists $\alpha > 0$ such that for any finite set $\Phi \subset G$, for any $\epsilon > 0$ there exists a finite group H with a bi-invariant metric d and a (Φ, ϵ, α)-homomorphism to (H, d).*

Remark 4.2 • *In Definition 4.1 we do not ask the metric to be normalized. So, α may be any fixed positive number.*
• *It is easy to see, that any sofic group is w-sofic.*
• *The idea of using bi-invariant metrics in the context of sofic groups appears in [17]. We also discussed it with E. Gordon.*

Theorem 4.3 *Let F be a finitely generated free group and $N \lhd F$. Then F/N is w-sofic if and only if for any finite sequence $g_1, g_2, ..., g_k$ from N one has $\overline{[g_1]^F [g_2]^F ... [g_k]^F} \subseteq N$.*

($\overline{X}$ denote the closure of X in the profinite topology on F.)

Corollary 4.4 *If there exists a non w-sofic group then there exists a finitely presented non w-sofic group.*

PROOF. It follows from the definition of w-sofic groups that if there exists a non w-sofic group then there exists a finitely generated non w-sofic group G. So, $G = F/N$ for a free group $F = \langle x_1, x_2, ..., x_n \rangle$. Then there exist $g_1, g_2, ..., g_k \in N$ such that $\overline{[g_1]^F [g_2]^F ... [g_k]^F} \not\subset N$. Now, $N(g_1, g_2, ..., g_k) \subseteq N$, where $N(g_1, g_2, ..., g_k)$ is the normal subgroup generated by $g_1, g_2, ... g_k$. So, $\overline{[g_1]^F [g_2]^F ... [g_k]^F} \not\subset N(g_1, g_2, ..., g_k)$ and the group $\langle x_1, x_2, ..., x_n \mid g_1, g_2, ..., g_k \rangle$ is not w-sofic. $\quad\square$

Conjecture 4.5 *There exists a non w-sofic group.*

Corollary 4.6 *Conjecture 4.5 and Conjecture 2.1 are equivalent.*

To finish this section we present another point of view on Theorem 4.3. Let F be a free group, $N \lhd F$ and $\tilde{F}$ be its profinite completion. It is clear that $F < \tilde{F}$ and $N < \tilde{F}$, but in general $N \not\lhd \tilde{F}$. Let $\hat{N}$ denote the minimal normal subgroup such that $N < \hat{N} \lhd \tilde{F}$ and $\tilde{N}$ denote the closure of N in $\tilde{F}$. It is easy to see that $\hat{N} \le \tilde{N} \lhd \tilde{F}$ and $\hat{N} = \tilde{N}$ iff $\hat{N}$ is closed in $\tilde{F}$. The following corollary is a consequence of Theorem 4.3 (with known facts about residually finite groups).

Corollary 4.7 *The group F/N is w-sofic iff $N = \hat{N} \cap F$ and the group F/N is residually finite iff $N = \tilde{N} \cap F$.*

Remark 4.8 *Let $S = [g_1]^{\tilde{F}}[g_1^{-1}]^{\tilde{F}}[g_2]^{\tilde{F}}[g_2^{-1}]^{\tilde{F}} \cdots [g_k]^{\tilde{F}}[g_k^{-1}]^{\tilde{F}}$. Then $\hat{N}(g_1, g_2, ..., g_k) = \bigcup_{n=1}^{\infty} S^n$. So, $\hat{N}(g_1, g_2, ..., g_k)$ is a closed set if and only if $\hat{N}(g_1, g_2, ..., g_k) = S^n$ for some n, due to the fact that S is a compact set, see [10].*

5 Bi-invariant metrics on finite groups

Any bi-invariant metrics $d : G \times G \to \mathbb{R}$ on a group G may be defined as $d(a, b) = \|ab^{-1}\|$ where the "norm" $\|g\| = d(e_G, g)$ satisfies the following properties ($\forall g, h \in G$):

(1) $\|g\| \ge 0$
(2) $\|e_G\| = 0$
(3) $\|g^{-1}\| = \|g\|$
(4) $\|hgh^{-1}\| = \|g\|$ ($\|\cdot\|$ is a function of conjugacy classes)
(5) $\|gh\| \le \|g\| + \|h\|$ (it is, in fact, the triangle inequality for d.)

Remark 5.1 *In fact, properties 1-5 define only semimetric. However, $N = \{g \in G : \|g\| = 0\}$ is a normal subgroup. Since $\|\cdot\|$ is constant on left (right) classes of N, it naturally defines metric on G/N.*

Let us give examples of such metrics:

I. The normalized Hamming metric on S_n $\|x\| = \frac{|\{j \mid j \ne jx\}|}{n}$.

II. One can use bi-invariant matrix metrics on a representation of a group G. For example if one takes the normalized trace norm (the normalized Hilbert-Schmidt norm) on an unitary representations with character χ one gets:

$$\|g\|_\chi = \sqrt{\frac{2\chi(e_G) - (\chi(g) + \chi^*(g))}{\chi(e_G)}}.$$

(If χ is the fixed point character on S_n, then $\|g\|_\chi = \sqrt{2h_n(e_{S_n}, g)}$.)

III. The following construction will be used in the proof of Theorem 4.3. We will use "conjugacy class graph" which is an analogue of Cayley graph,

where conjugacy classes are used instead of group elements. Let G be a (finite) group and CG be the set of all its conjugacy classes, let $\mathcal{C} \subset CG$. The conjugacy graph $\Gamma(G, \mathcal{C})$ is defined as follows: its vertex set $V = CG$ and for $x, y \in V$ there is an edge (x, y) iff $x \subset cy$ for some $c \in \mathcal{C}$. (The graph $\Gamma(G, \mathcal{C})$ is considered as an undirected graph.) See fig.1 for example.
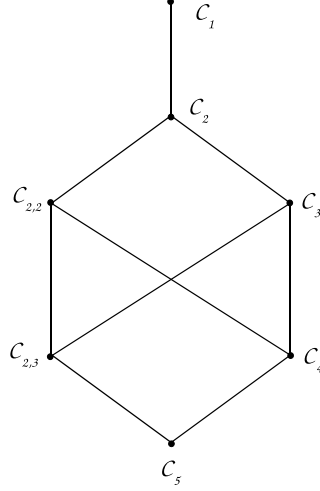

Fig. 1. The conjugacy graph $\Gamma(S_5, \{\mathcal{C}_2\})$, where the vertices $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_5, \mathcal{C}_{2,2}, \mathcal{C}_{2,3}$ are the conjugacy classes of $e_{S_5}, (1, 2), (1, 2, 3), (1, 2, 3, 4), (1, 2, 3, 4, 5), (1, 2)(3, 4), (1, 2)(3, 4, 5)$ respectively.

Now define $\|g\|_{\mathcal{C}}$ to be the distance from $\{e_G\}$ to $[g]^G$ in $\Gamma(G, \mathcal{C})$, if $[g]^G \in K(e_G)$, the connected component of $\{e_G\}$. For $[g]^G \notin K(e_G)$ let $\|g\| = \max\{\|x\| \mid x \in K(e_G)\}$. One can check that $\|\cdot\|_{\mathcal{C}}$ satisfies 1-5 by construction.

6 Proof of Theorem 4.3

Theorem 4.3 is a direct consequence of the following lemmata.

We will represent the elements of a free group F by the reduced words. For $w \in F$ let $|w|$ denotes the length of w.

Definition 6.1 *Let F be a free group, $N \triangleleft F$, $\delta > \epsilon > 0$ and $r \in \mathbb{N}$. Let H be a finite group with a bi-invariant metric d and $\phi : F \to H$ be a homomorphism. We will say that N is (r, ϵ, δ)-separated by ϕ if for any $w \in F$, $|w| \leq r$ we have the following alternative:*

- *$d(e_H, \phi(w)) < \epsilon$ if $w \in N$,*
- *$d(e_H, \phi(w)) > \delta$ if $w \notin N$.*

In this case we call N to be (r, ϵ, δ)-separable.

We will say that a normal subgroup N is finitely separable if there exists $\delta > 0$ such that for any $r \in \mathbb{N}$ any $\epsilon > 0$ the normal subgroup N is (r, ϵ, δ)-separable.

Lemma 6.2 *Let F be a finitely generated free group. Then $G = F/N$ is w-sofic if and only if N is finitely separable.*

PROOF. Let us give two proofs of the lemma: the first one by using nonstandard analysis and the second without it. A short introduction to non standard analysis in a similar context could be found in [1,17]. Our use of nonstandard analysis is not essential: one can easily rewrite the proof on standard language (without using ultrafilters), on the other hand the non-standard proof is more algebraic and based on the following simple

Claim 6.3 *Let F be a free group and H be a group, $N \lhd F$ and $\mathcal{E} \lhd H$. Then the following are equivalent:*

(1) There exists a homomorphism $\phi : F \to H$ such that $\phi(N) \subseteq \mathcal{E}$ and $\phi(F \setminus N) \cap \mathcal{E} = \emptyset$.
(2) There exists an injective homomorphism $\tilde{\phi} : F/N \to H/\mathcal{E}$.

PROOF. Item 1 $\Rightarrow$ item 2. Let $\tilde{\phi}([w]_N) = [\phi(w)]_{\mathcal{E}}$. It is well-defined and injective by item 1.
Item 2 $\Rightarrow$ item 1. Let $F = \langle x_1, x_2, ..., x_n \rangle$. Let $[y_i]_{\mathcal{E}} = \tilde{\phi}([x_i]_N)$. Define $\phi : F \to H$ by setting $\phi(x_i) = y_i$ (F is free!). ϕ depends on the choice of y_i, but for all $w \in F$ one has $[\phi(w)]_{\mathcal{E}} = \tilde{\phi}([w]_N)$, (induction on $|w|$). It proves the claim. $\quad \square$

Non-standard proof. Let H be a hyperfinite group with an internal bi-invariant metric d, then $\mathcal{E} = \{h \in H \mid d(h, e) \approx 0\}$ is a normal subgroup (Since d is bi-invariant). Group $G = F/N$ is w-sofic iff there exists a homomorphic injection $\tilde{\phi} : G \to H/\mathcal{E}$. On the other hand, N is finitely separated iff there exists a homomorphism $\phi : F \to H$, such that $\phi(N) \subseteq \mathcal{E}$ and $\phi(F \setminus N) \cap \mathcal{E} = \emptyset$, this is equivalent to the existence of a homomorphic injection $\tilde{\phi} : F/N \to H/\mathcal{E}$ by the claim. The lemma follows.

Standard proof.

$\Longrightarrow$ Let $G = F/N$ and $F = \langle x_1, x_2, ..., x_n \rangle$, let $\Phi = \{[w]_N \mid w \in F, |w| \leq r\}$ and $\phi : \Phi \to H$ be a (Φ, ϵ, α)-homomorphism ($[w]_N = wN$ denotes the right N-class of w). Define a homomorphism $\tilde{\phi} : F \to H$ by setting $\tilde{\phi}(x_i) = \phi([x_i]_N)$. It is enough to show that $\tilde{\phi}$ will $(r, 2r\epsilon, \alpha - 2r\epsilon)$-separate N. Using the inequality

$d(\phi([x_i]_N)^{-1}, \phi([x_i^{-1}]_N)) < \epsilon$ and induction on $|w|$ one gets

$$d(\tilde{\phi}(w), \phi([w]_N)) < (2|w| - 1)\epsilon.$$

So, if $w \in N \cap \{w \in F : |w| \le r\}$, then $[w]_N = e_G$ and

$$d(\tilde{\phi}(w), e_H) < (2|w| - 1)\epsilon < 2r\epsilon,$$

if $w \notin N$ then

$$d(\tilde{\phi}(w), e_H) \ge d(\phi([w]_N), e_H) - d(\tilde{\phi}(w), \phi([w]_N)) > \alpha - 2r\epsilon.$$

$\Longleftarrow$ We have to construct a (Φ, ϵ, α)-homomorphism. Let $\Phi = \{[w_1]_N, [w_2]_N, ..., [w_m]_N\}$, such that e_G (if exists) is presented by the empty word, let $r = \max\{|w_1|, ..., |w_m|\}$. Choose ϕ to $(3r, \epsilon, \alpha)$-separate N, and define $\tilde{\phi} : \Phi \to H$ as $\tilde{\phi}([w_i]_N) = \phi(w_i)$. (We suppose, that all $[w_i]_N$ are different, so $\tilde{\phi}$ is well-defined, although depends on the choice of w_i.) We claim that $\tilde{\phi}$ is a (Φ, ϵ, α)-homomorphism. Indeed,

- $\tilde{\phi}(e_G) = e_H$,(e_G is represented by the empty word)
- $d(\tilde{\phi}([w_i]_N, e_H) = d(\phi(w_i), e_H) > \alpha$
- if $[w_i]_N[w_j]_N = [w_k]_N$ then

$$d(\tilde{\phi}([w_i])\tilde{\phi}([w_j]), \tilde{\phi}([w_k])) = d(\phi(w_i)\phi(w_j), \phi(w_k)) = d(\phi(w_i w_j w_k^{-1}), e_H) < \epsilon,$$

since $w_i w_j w_k^{-1} \in N$ and has length $\le 3r$. $\quad\square$

Lemma 6.4 *Let F be a finitely generated free group and N be its normal subgroup. Then N is finitely separable if and only if for any $g_1, g_2, ..., g_k \in N$ one has $\overline{[g_1]^F[g_2]^F...[g_k]^F} \subseteq N$.*

PROOF. $\Longrightarrow$ In order to prove that $\overline{[g_1]^F[g_2]^F...[g_k]^F} \subseteq N$ it is enough to prove that for any $w \notin N$ there exists a homomorphism $\phi : F \to H$ into a finite group H such that $\phi(w) \notin [\phi(g_1)]^H[\phi(g_2)]^H...[\phi(g_k)]^H$.

Let $r > \max\{|w|, |g_1|, |g_2|, ..., |g_k|\}$. Take $\phi : F \to H$ which $(r, \alpha/k, \alpha)$-separates N, then $d(e_H, \phi(w)) > \alpha$ and $d(e_H, \phi(g_i)) < \alpha/k$. But if $\phi(w) \in [\phi(g_1)]^H[\phi(g_2)]^H...[\phi(g_k)]^H$, then (by 4 and 5 of Section 5)

$$d(e_H, \phi(w)) \le \sum_{i=1}^{k} d(e_H, \phi(g_i)) < \alpha,$$

a contradiction.

$\Longleftarrow$ We have to construct an (r, ϵ, α)-separating homomorphism. Let $W_r = \{w \in F, : |w| \le r\}$ and integer $k > \alpha/\epsilon$. We can find a homomorphism

$\phi : F \rightarrow H$ into a finite group H, such that for any $w \in W_r \setminus N$ and any $g_1, g_2, ..., g_m \in W_r \cap N$, $m \leq k$ one has $\phi(w) \notin [\phi(g_1)]^H [\phi(g_2)]^H ... [\phi(g_m)]^H$. To define a metric on H let $\mathcal{C} = \{[\phi(g)]^H \mid g \in W_r \cap N\}$ and now let $d(x, e) = \epsilon \|x\|_{\mathcal{C}}$, where $\| \cdot \|_{\mathcal{C}}$ is defined in item III of section 5. Now one can check that $\phi : F \rightarrow H$ will (r, ϵ', α)-separate N, for any $\epsilon' > \epsilon$. $\square$

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